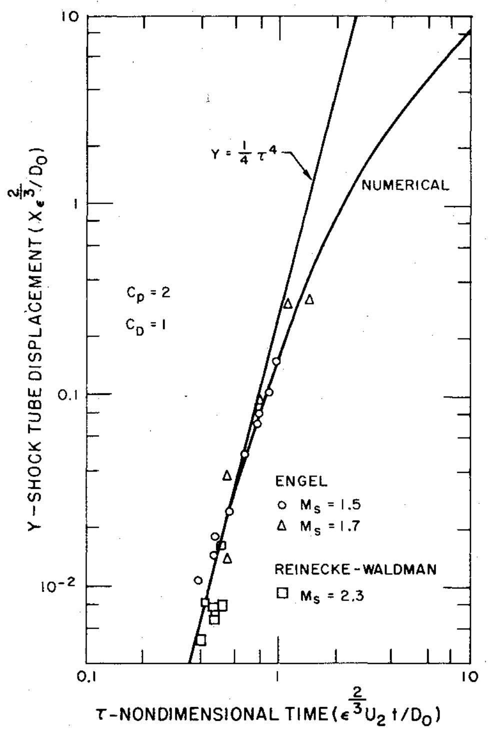

Fig. 3 Late time solution for the shock tube displacement.

τ^2 near $\tau=2$ and a linear behavior is noted for $\tau \geq 10$. These results appear to explain why investigators have been unable to correlate data with a single power law.

Conclusions

A simplified theory for liquid drop acceleration and deformation has been developed and compared to data. The results illustrate that the displacement increases as τ^4 at early time and as τ at late time. Hence, no single power law can correlate all of the data, although they may be good approximations over narrow ranges of τ.

Empirical results have previously been presented in nondimensional length and time scales which correspond to rigid particle acceleration. Reference variables for deformable particles have a different ϵ dependence than those for rigid particles. Hence, empirical results should be correlated in the reference variables appropriate to deformable particles.

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Large Amplitude Free Vibrations of Tapered Beams

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Nomenclature

a	= maximum amplitude
A	= area of cross-section of the beam at any x
A_c	= area of cross-section at $x=0$
b, d	= breadth and depth of the beam, respectively, at any x
b_c, d_c	= breadth and depth at $x=0$, respectively
b_e, d_e	= breadth and depth at $x=\pm l$, respectively
E	= Young's modulus
I	= moment of inertia at any x
I_c	= moment of inertia at $x=0$
l	= length of the beam
N_x	= axial force developed in the beam due to large deformation
r_c	= radius of gyration at $x=0$; given by $(I_c/A_c)^{1/2}$
u	= axial displacement
w	= transverse displacement
x	= axial coordinate measured from the center of the beam
α	= taper parameter; defined as $(b_c-b_e)/b_c$ for a beam with breadth taper and $(d_c-d_e)/d_c$ for a beam with depth taper
β	= a/r_c
ρ	= mass density
ξ	= nondimensional coordinate $=2x/l$
ω_{NL}, ω_L	= nonlinear and linear circular frequencies, respectively
$(\ )'$	= denotes differentiation with respect to the axial coordinate x
$(\dot{\ })$	= denotes differentiation with respect to time t

Introduction

LARGE amplitude free flexural vibrations of uniform-beams were studied, using continuum[1,2] and finite element methods.[3,4] Woinowsky-Krieger[1] obtained an elliptic integral solution, Srinivasan[2] used Ritz-Galerkin method, Chuh-Mei[3] and Venkateswara Rao et al.[4] used two different finite element formulations to obtain frequency-amplitude relationships for the fundamental mode.

In this Note, large amplitude vibrations of simply-supported and clamped tapered beams of rectangular cross-section, which are often encountered in practical structures, are investigated using Galerkin method. Two types of linear tapers, namely, breadth and depth tapers are considered in this study. For each type of taper, two solutions are obtained

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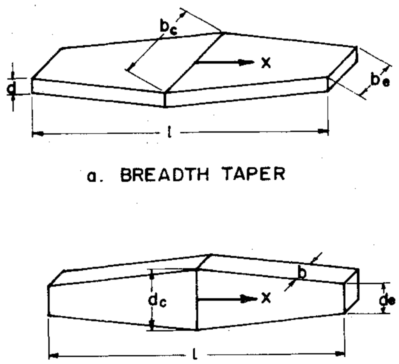

a. BREADTH TAPER

b. DEPTH TAPER

Fig. 1 Geometry of tapered beams.

using trigonometric and polynomial displacement distributions. The frequency-amplitude relationships are obtained for all those cases, for the fundamental flexural mode. Solutions due to Srinivasan[2] for uniform beams can be obtained as degenerated cases of the present solutions.

Formulation

Figures 1a and 1b give the configurations of two types of tapered beams considered in this Note. The governing differential equation of such beams executing large amplitude flexural oscillations is

$$(EIw'')'' + \rho A \ddot{w} - N_x w'' = 0 \tag{1}$$

N_x is the axial force developed in the beam due to large amplitudes and is assumed to be constant as in the earlier investigations. To evaluate this, the stretching of the beam due to large amplitudes is calculated as

$$u = \tfrac{1}{2} \int_{-l/2}^{l/2} (w')^2 \mathrm{d}x \tag{2}$$

The stress σ_x developed to prevent this stretching at any section x is

$$\sigma_x = \frac{N_x}{A} = Eu' \tag{3}$$

Integrating Eq. (3) one has

$$u = \frac{N_x}{E} \int_{-l/2}^{l/2} \frac{\mathrm{d}x}{A} \tag{4}$$

Equations (2 and 4) give

$$N_x = \frac{E}{2} \frac{\int_{-l/2}^{l/2} (w')^2 \mathrm{d}x}{\int_{-l/2}^{l/2} \frac{\mathrm{d}x}{A}} \tag{5}$$

For uniform beams, this expression for N_x degenerates to the ones given in Refs. 1-3.

To apply the Galerkin method, for a beam executing harmonic oscillations, assume

$$w = a\phi(x) \sin \omega t \tag{6}$$

where $\phi(x)$ satisfies all boundary and symmetry conditions, and a is a constant.

The assumed displacement distribution of Eq. (6), in general, does not satisfy the governing differential equation. Let R be the error in the differential equation. This error is minimized in the Galerkin method as

$$\int_0^{2\pi/\omega} \int_{-l/2}^{l/2} R\phi(x) \sin \omega t \, \mathrm{d}x \, \mathrm{d}t = 0 \tag{7}$$

Equation (7) gives the frequency-amplitude relationships of a tapered beam.

Simply Supported Beam with Depth Taper

Consider a simply supported beam with depth taper and assume

$$w = a \cos \frac{\pi \xi}{2} \sin \omega t \tag{8}$$

where ξ is a nondimensional coordinate, $\xi = 2x/l$.

$$\text{For } 0 \le \xi \le 1;\ I = I_c(1-\alpha\xi)^3;\ A = A_c(1-\alpha\xi) \tag{9}$$

$$\text{and for } -1 \le \xi \le 0;\ I = I_c(1+\alpha\xi)^3;\ A = A_c(1+\alpha\xi) \tag{10}$$

Equation (8) satisfies all boundary conditions and the symmetry conditions at the center of the beam except,

$$(EI\,w'')' = 0 \quad \text{at} \quad \xi = 0 \tag{11}$$

This will leave a residual error ϵ_s. This error is also normalized and added to Eq. (7). Equation (7) can, then, be rewritten as

$$\int_0^{2\pi/\omega} \int_{-l/2}^{l/2} R\phi(x) \sin \omega t \, \mathrm{d}x \, \mathrm{d}t + R_s = 0 \tag{12}$$

where

$$R_s = \epsilon_s \cdot \omega_{\xi=0} \tag{13}$$

Substituting the distribution for w of Eq. (8) into Eqs. (5, 12 and 13) one gets

$$1 - \frac{\omega_{NL}^2}{\omega_L^2} - \frac{a^2}{r_c^2} \frac{3}{16} \frac{1}{1+\lambda} \frac{\alpha}{Ln(1-\alpha)} = 0 \tag{14}$$

where

$$\frac{\rho A_c \omega_L^2 l^4}{EI_c} = \pi^4 \frac{1+\lambda}{1+\mu} \tag{15}$$

with

$$\lambda = \alpha\left(\frac{6}{\pi^2} - \frac{3}{2}\right) + \alpha^2\left(1 - \frac{6}{\pi^2}\right) + \alpha^3\left(-\frac{12}{\pi^4} - \frac{1}{4} + \frac{3}{\pi^2}\right) \tag{16}$$

and

$$\mu = \left(\frac{2}{\pi^2} - \frac{1}{2}\right)\alpha$$

Equation 14 gives the frequency-amplitude relationship of simply supported beam with depth taper. When the beam is uniform Eq. (14) degenerates to

$$1 - \frac{\omega_{NL}^2}{\omega_L^2} + \frac{3}{16} \frac{a^2}{r^2} = 0 \tag{17}$$

which is the same as given in Ref. 2.

Table 1 Tapered beams: frequency amplitude relationships

Beam	Assumed displacement distribution for w	ω_{NL}^2/ω_L^2	Linear frequency parameter $\rho A_c \omega_L^2 l^4/E I_c$	Breadth taper λ	Breadth taper μ	Depth taper λ	Depth taper μ
Simply supported	$a\cos\frac{\pi\xi}{2}$	$1-\frac{3}{16}\beta^2\frac{\alpha}{Ln(1-\alpha)}\frac{1}{1+\lambda}$	$\pi^4\frac{1+\lambda}{1+\mu}$	$(\frac{2}{\pi^2}-\frac{1}{2})\alpha$	$(\frac{2}{\pi^2}-\frac{1}{2})\alpha$	$\alpha(\frac{6}{\pi^2}-\frac{3}{2})+\alpha^2(1-\frac{6}{\pi^2})+\alpha^3(-\frac{12}{\pi^4}-\frac{1}{4}+\frac{3}{\pi^2})$	$(\frac{2}{\pi^2}-\frac{1}{2})\alpha$
	$a(1-\frac{6}{5}\xi^2+\frac{1}{5}\xi^4)$	$1-\frac{3}{16}\beta^2\frac{\alpha}{Ln(1-\alpha)}\cdot\frac{1}{1+\lambda}\frac{1088\times 17}{875\times 21}$	$\frac{48\times 63}{31}\frac{1+\lambda}{1+\mu}$	$-\frac{5}{16}\alpha$	$-\frac{113\times 21}{7936}\alpha$	$-\frac{15}{6}\alpha+\frac{3}{7}\alpha^2-\frac{5}{64}\alpha^3$	$-\frac{113\times 21}{7936}\alpha$
Clamped	$\frac{a}{2}(1+\cos\pi\xi)$	$1-\frac{3}{16}\beta^2\frac{\alpha}{Ln(1-\alpha)}\cdot\frac{1}{4(1+\lambda)}$	$\frac{16\pi^4}{3}\cdot\frac{1+\lambda}{1+\mu}$	$-\frac{\alpha}{2}$	$(\frac{8}{3\pi^2}-\frac{1}{2})\alpha$	$-\frac{3}{2}\alpha+(1+\frac{3}{2\pi^2})\alpha^2-\alpha^3(\frac{1}{4}+\frac{3}{4\pi^2})$	$(\frac{8}{3\pi^2}-\frac{1}{2})\alpha$
	$a(\xi^2-1)^2$	$1-\frac{3}{16}\beta^2\frac{\alpha}{Ln(1-\alpha)}\cdot\frac{1}{(1+\lambda)}\cdot\frac{512}{105\times 21}$	$504\frac{1+\lambda}{1+\mu}$	$-\frac{5}{8}\alpha$	$-\frac{63}{256}\alpha$	$-\frac{15}{8}\alpha+\frac{11}{7}\alpha^2-\frac{15}{32}\alpha^3$	$-\frac{63}{256}\alpha$

Table 2 ω_{NL}/ω_L of a tapered beam for $\alpha=0.4$

Amplitude ratio $\beta=a/r_c$	Simply supported beam				Clamped beam			
	Breadth taper		Breadth taper		Breadth taper		Depth	
	trigonometric[a]	polynomial[b]	trigonometric[a]	polynomial[b]	trigonometric[c]	polynomial[d]	trigonometric[c]	polynomial[d]
0.1	1.0008	1.0008	1.0010	1.0010	1.0002	1.0002	1.0004	1.0005
0.2	1.0033	1.0034	1.0042	1.0042	1.0009	1.0009	1.0017	1.0018
0.4	1.0132	1.0134	1.0166	1.0169	1.0037	1.0036	1.0066	1.0074
0.6	1.0296	1.0300	1.0370	1.0375	1.0082	1.0081	1.0149	1.0165
0.8	1.0520	1.0527	1.0649	1.0658	1.0146	1.0144	1.0263	1.0291
1.0	1.0801	1.0812	1.0997	1.1011	1.0227	1.0225	1.0408	1.0451
2.0	1.2910	1.2944	1.3354	1.3601	1.0879	1.0871	1.1545	1.1702
3.0	1.5811	1.5875	1.6981	1.7065	1.1887	1.1870	1.3224	1.3532

[a] $w=a\cos\pi\xi/2$.
[b] $w=a(1-6/5\,\xi^2+1/5\,\xi^4)$.
[c] $w=a/2\,(1+\cos\pi\xi)$.
[d] $w=a(\xi^2-1)^2$.

The previous analysis is repeated for a simply supported beam for a polynomial distribution and a clamped beam with trigonometric and polynomial distributions for breadth and depth tapers. The results are summarized in Table 1.

Numerical Results

Table 2 presents ω_{NL}/ω_L for simply supported and clamped beams with breadth and depth tapers for a typical taper parameter $\alpha=0.4$. These results are obtained with the solutions presented in Table 1. Table 2 demonstrates good agreement between trigonometric and polynomial solutions for all cases. Further, it is noted that the nonlinearity is always of hardening type and, as expected, the nonlinearity is severe for beams with depth taper compared to beams with breadth taper.

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Anisotropy and Shear Deformation in Laminated Composite Plates

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Introduction

IT has long been recognized that transverse shear flexibility and material anisotropy (nonorthotropy) can play an important role in reducing the effective stiffness of laminated composite plates. However, most of the published results for statically loaded plates are for simply supported edges and are limited to comparisons of the maximum deflections and stresses obtained by refined theories (which include transverse shear flexibility) with those obtained by classical laminated plate theory.[1-3] To the authors' knowledge, no quantitative measures have been used which account for the effects of the different lamination and geometric parameters on the transverse shear flexibility and material anisotropy of composite plates. The present Note summarizes some of the results of a

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